

NON-SPLIT LINEAR SHARPLY 2-TRANSITIVE GROUPS, AFTER RIPS-SEGEV-TENT

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ABSTRACT. We construct examples of countable linear groups $\Gamma < \mathrm{SL}_n(\mathbf{R})$ with no non-trivial normal abelian subgroup that admit a faithful, sharply 2-transitive action on a set. The stabilizer of a point in this action does not contain an involution.

1. INTRODUCTION

A sharply 2-transitive group is, by definition, a permutation group $\Gamma \curvearrowright X$ which acts transitively and freely on ordered pairs of distinct points. Such a group is called *split* if it admits a non-trivial normal abelian subgroup. The following question, that has attracted the attention of algebraists for many years, was recently answered negatively, by Rips Segev and Tent in [RST]

Question 1.1. *Is every sharply 2-transitive group split?*

In Theorem 1.4 we show that the answer remains negative even in the setting of countable linear groups. This contrasts nicely with the prior results of [GG14, GMS15] that show that the answer to the same question is positive for linear groups when the permutational characteristic of Γ is not 2; or in other words under the additional assumption that involutions in Γ fix a point.

Splitting implies a tame, algebraic, structure theory. In particular with every split sharply 2-transitive group Γ one can associate a near field N , which is by definition a division ring that is distributive only from the right¹. In this case Γ indeed *splits* as a semi-direct product $\Gamma = N^* \ltimes N$ of the multiplicative and additive groups of the near field. Moreover the given sharply 2-transitive action is the unique faithful primitive permutation representation of this group and it is isomorphic to the natural action $N^* \ltimes N \curvearrowright N$ by affine transformation $x^{(a,b)} = x \cdot a + b$.

When $\Gamma \curvearrowright X$ is sharply 2-transitive there exists an element flipping any two points in X , whose square must be trivial. This gives rise to a large set of involutions $\mathrm{Inv}(\Gamma) \subset \Gamma$. Such a pair of points determines the involution and since Γ is transitive on pairs of

Date: August 16, 2016.

2010 Mathematics Subject Classification. Primary 20B22.

Key words and phrases. sharply 2-transitive, linear groups.

¹Following [RST] all the actions in this paper will be right actions. This choice is also responsible for the choice of right near fields here.

points all involutions are conjugate. Any nontrivial element, and in particular any involution, can have either 0 or 1 fixed points. If an involution does fix a point then the map $\text{Inv}(\Gamma) \rightarrow X$ taking an involution to its fixed point is a Γ invariant bijection. Consequently the Γ action on $\text{Inv}(\Gamma)$ by conjugation is 2-transitive and in particular, the order of the product of two different involutions is independent of the choice of the specific involutions. This gives rise to the following definition:

Definition 1.2. *Let Γ be a sharply 2-transitive permutation group on X . If the stabilizer of a point contains an involution let $p = \text{Ord}(\sigma\tau)$ be the order of the product of two distinct involutions. We define the permutational characteristic of Γ to be*

$$p\text{-char}(\Gamma) = \begin{cases} 2 & \text{Involutions do not fix a point} \\ p & p < \infty \\ 0 & p = \infty \end{cases}$$

It is not difficult to verify that $p\text{-char}(\Gamma)$ is either 0 or prime, and that it coincides with the characteristic of the near field whenever Γ splits. We refer the readers to [Ker74], [GG14] for more details.

In two papers [Zas35a, Zas35b] from 1936 H. Zassenhaus completed a full classification of finite sharply 2-transitive groups. He started by showing that every such group splits, and then gave a complete classification of finite near fields. Contrary to the situation for skew fields, the latter classification involves non-trivial examples of finite near fields. In [Tit52a, Tit52b] Tits proved that every locally compact connected sharply 2-transitive group splits. The near fields here are just \mathbf{R} , \mathbf{C} or \mathbf{H} . In [Tür04] Türkelli proved that every sharply 2-transitive group Γ with $p\text{-char}(\Gamma) = 3$ splits. In [GG14, GMS15] it was shown that every linear sharply 2-transitive group $\Gamma < \text{GL}_n(k)$ with $p\text{-char}(\Gamma) \neq 2$ splits. This was shown by us under the additional assumption that $\text{char}(k) \neq 2$, an assumption that was deemed unnecessary by [GMS15] who also relaxed the linearity assumption.

Definition 1.3. *Let $\Delta \curvearrowright Y$, $\Gamma \curvearrowright X$ be two actions of groups on sets. An embedding of such actions $(\phi, \iota) : (\Delta \curvearrowright Y) \hookrightarrow (\Gamma \curvearrowright X)$ is a monomorphism $\phi : \Delta \rightarrow \Gamma$ and a ϕ -equivariant injective map $\iota : Y \rightarrow X$.*

Such an embedding is obtained, in particular, whenever $\Delta < \Gamma$ is a subgroup and there is a Δ -orbit in Y that is permutation isomorphic to the given action of Δ on X . At the group theoretic level if the two actions are identified with coset actions $Y = A \backslash \Delta$, $X = B \backslash \Gamma$ then we are just looking at an embedding $\phi : \Delta < \Gamma$ such that $A = \Delta \cap B$.

Recently the first examples of non-split sharply 2-transitive groups were given by Rips-Segev-Tent in [RST]. Their examples are very general in the sense that they show that every transitive group action on a set $G \curvearrowright X$ with the properties that the stabilizer of

a point does not contain an involution and the stabilizer of every pair of distinct points is trivial can be embedded into a non-split sharply 2-transitive action of permutational characteristic 2. Clearly these two conditions are also necessary for such an embedding.

Our main theorem comes to show that such non-split sharply 2-transitive groups of permutational characteristic 2 can be constructed even within the realm of linear groups. Our main theorem is:

Theorem 1.4. (*Linear s-2-t groups*) *Let $H < \mathrm{SL}_n(\mathbf{R})$ be a countable group that contains neither involutions nor nontrivial scalar matrices. Assume that $H \curvearrowright X$ is a transitive permutation action with the property that the stabilizer of every pair of distinct points is trivial. Then there exists a larger countable group $H < H_1 < \mathrm{SL}_n(\mathbf{R})$ which admits a sharply 2-transitive non-split permutation representation $H_1 \curvearrowright X_1$ of permutational characteristic 2 and an embedding:*

$$(H \curvearrowright X) \hookrightarrow (H_1 \curvearrowright X_1).$$

In some examples the resulting group H_1 admits uncountably many non-isomorphic faithful primitive permutation representations $H_1 \stackrel{\rho_\alpha}{\curvearrowright} X_\alpha$.

Remark 1.5. *The last statement of the theorem should be contrasted with the well known fact that a split sharply 2-transitive group admits a unique (up to isomorphism of permutations representations) faithful primitive permutation representation. Thus the nonsplit examples of [RST] are the first natural candidates for sharply 2-transitive groups that admit multiple faithful primitive actions. In order to actually construct such examples linearity comes in handy. We appeal to the results of [GG13] which ensure that any Zariski dense countable subgroup of $\mathrm{SL}_n(\mathbf{R})$ with trivial center admits uncountably many non-isomorphic faithful primitive actions. But linearity is not essential here. Using the methods of [RST], it is possible to construct groups that admit many faithful non-isomorphic sharply 2-transitive actions! This will be shown in subsequent paper.*

2. REDUCTIONS

Given an involution $t \in \mathrm{SL}_n(\mathbf{R})$ let $\mathbf{R}^n = W^+(t) \oplus W^-(t)$ denote the decomposition of \mathbf{R}^n into its ± 1 eigenspaces and $\pi_t = \frac{1+t}{2} : \mathbf{R}^n \rightarrow W^+(t)$ the projection on $W^+(t)$ along $W^-(t)$. When possible, we will omit t from the notation writing W^\pm instead of $W^\pm(t)$. Our main technical theorem is the following, linear version of [RST, Theorem 1.1]:

Theorem 2.1. *Let $G < \mathrm{SL}_n(\mathbf{R})$ be a countable group, $t \in G$ an involution and $A < G$ a malnormal subgroup containing no involutions. Assume further that all the involutions in G are conjugate (in G), that $\dim(W^+(t)) \geq \dim(W^-(t)) + 2$, and whenever $W^+(t)$ is contained in an eigenspace of $\pi_t \circ g$ for some $g \in G$ then either $g = 1$ or $g = t$.*

Then for any element $v \in G \setminus A$ there exists a countable extension $G \leq G_1 < \mathrm{SL}_n(\mathbf{R})$ with a malnormal subgroup $A_1 < G_1$ containing no involutions, such that $A_1 \cap G = A$ and an element $f \in A_1$ such that $A_1 t f = A_1 v$. Moreover all the involutions in G_1 are conjugate and the only elements of G_1 such that $W^+(t)$ is contained in an eigenspace of $\pi_t \circ g$ are $g = 1$ and $g = t$.

The significant difference between this theorem and [RST, Theorem 1.1] is the linearity requirement $G, G_1 < \mathrm{SL}_n(\mathbf{R})$. As we plan to use this theorem infinitely many times within an induction argument, it is particularly important for us that G_1 is realized as a linear group within the same ambient matrix group as the original group G . A few “auxiliary requirements” become necessary in order to prove the theorem in the linear setting: That G be countable, with conjugate involutions and that if $g \notin \{1, t\}$ then $W^+(t)$ is not contained in an eigenspace of $\pi_t \circ g$.

In the proof we follow the strategy of [RST]. For the convenience of the reader we quote here three of their propositions. The first proposition reduces the main theorem to two special cases.

Proposition 2.2. *It is enough to prove Theorem 2.1 under the additional assumption that $v, v^{-1} \notin AtA$ and either $v^{-1} \notin AvA$ or v is an involution.*

Proof. See [RST, Section 2], the exact same reduction works for us here. \square

The next two theorems treat these two cases respectively:

Theorem 2.3. ([RST, Theorem 3.1]) *Let G be a group, $A < G$ a malnormal subgroup containing no involutions, $t, v \in G$ two elements with t an involution and $v, v^{-1} \notin AtA$ and $v^{-1} \notin AvA$. If A already contains an element f such that $Atf = Av$ then take $G_1 = G, A_1 = A, f = f$. Otherwise set*

$$G_1 := G * \langle f \rangle, \quad A_1 = \langle A, f, t f v^{-1} \rangle.$$

Where $\langle f \rangle = \mathbf{Z}$.

Then A_1 is malnormal in G_1 , $A_1 t f = A_1 v$ and $G \cap A_1 = A$.

Theorem 2.4. [RST, Theorem 4.1] *Let G be a group, $A < G$ a malnormal subgroup containing no involutions, $t, v \in G$ two involutions such that $v, v^{-1} \notin AtA$. If A already contains an element f such that $Atf = Av$ then take $G_1 = G, A_1 = A, f = f$. Otherwise set*

$$G_1 = \langle G, f \mid f^{-1} t f = v \rangle, \quad A_1 := \langle A, f \rangle.$$

Then A_1 is malnormal in G_1 , $A_1 t f = A_1 v$ and $G \cap A_1 = A$.

The contribution of the current paper is in showing that in the setting of the above two theorems; when the group $G < \mathrm{SL}_n(\mathbf{R})$ is countable linear, and subject to all of the auxiliary conditions described in Theorem 2.1, then G_1 is still linear. In fact G_1 is still

contained in the same ambient matrix group $\mathrm{SL}_n(\mathbf{R})$. This is achieved in Theorems 4.3 and 4.4 appearing in Section 4. In turn this observation is enough to carry out the full Rips-Segev-Tent construction inside the ambient matrix group.

3. SOME PROJECTIVE DYNAMICS

All our notation is taken from [BG03, Section 3] and we refer the readers to that paper for more details.

We use extensively the action $SL_n(\mathbf{R}) \curvearrowright \mathbb{P} := \mathbf{R}^n/R^*$ on the projective space. If $0 \neq v \in \mathbf{R}^n$, $\langle 0 \rangle \neq W < \mathbf{R}^n$ are a nontrivial vector and subspace in \mathbf{R}^n we denote the corresponding projective point, and subspace by $[v] \in \mathbb{P}$, $[W] < \mathbb{P}$. Fix a norm $\|\cdot\|$ on \mathbf{R}^n , this gives rise in to a norm on the exterior product $\mathbf{R}^n \wedge \mathbf{R}^n$ which is used in turn to endow the projective space $\mathbb{P} = \mathbf{R}^n/R^*$ with the metric

$$d([v], [w]) = \frac{\|v \wedge w\|}{\|v\| \cdot \|w\|}$$

We will denote the ϵ neighborhood of a set in this metric by $(\Omega)_\epsilon = \{x \in \mathbb{P} \mid d(x, \Omega) < \epsilon\}$.

Any nontrivial matrix $0 \neq M \in M_n(\mathbf{R})$ gives rise to a partially defined map $[M] : \mathbb{P} \setminus [\ker(M)] \rightarrow \mathbb{P}$. A projective point $[v] \in \mathbb{P}$ is moved by M if and only if v is not an eigenvector of M . Indeed $[M]$ is not defined on $[v]$ if and only if v is an eigenvector with eigenvalue 0 and $[v]$ is a fixed point of $[M]$ if and only if v is an eigenvector with a nonzero eigenvalue. For example our auxiliary condition in Theorem 2.1 requires that whenever $g \notin \{1, t\}$ then the matrix $\pi_t \circ g = \frac{g+tg}{2}$ should move at least one projective point in $[W^+(t)]$. If $M \in \mathrm{GL}_n(\mathbf{R})$ then the map $[M]$ is defined on the whole projective space and is in fact a bilipschitz homeomorphism with respect to the metric defined above.

Lemma 3.1. *Let $B \in \mathrm{PGL}_r(\mathbf{R})$. If $v_0, v_1, \dots, v_r < \mathbf{R}^r$ are $r+1$ vectors in general position that are all eigenvectors of B . then B is a scalar matrix.*

Proof. By definition, $r+1$ vectors in \mathbf{R}^r are in *general position* if any i of them span an i -dimensional subspace as long as $i \leq r$. By counting considerations there must be at least one eigenspace $V < \mathbf{R}^r$ of dimension l containing at least $l+1$ of the eigenvectors $\{v_0, v_1, \dots, v_r\}$. Since the vectors are in general position then $l = r$, so B has an eigenvalue with r -linearly independent eigenvectors and is hence a scalar. \square

4. LINEARITY PROOFS

Proposition 4.1. *Let $n \geq 3$ and $G, H < \mathrm{SL}_n(\mathbf{R})$ be two countable groups that contain no nontrivial scalar matrices. Let $\pi : \mathbf{R}^n \rightarrow W$ be a linear projection with $\dim(\mathrm{Im}(\pi)) - \dim(\ker(\pi)) \geq 2$. For $f \in \mathrm{SL}_n(\mathbf{R})$ let $\Phi_f : G * H \rightarrow \mathrm{SL}_n(\mathbf{R})$ be defined by $\Phi_f(g) = g, \forall g \in G$ and $\phi_f(h) = fhf^{-1}, \forall h \in H$.*

Then for a Baire generic choice of $f \in \mathrm{SL}_n(\mathbf{R})$ the map Φ_f is injective and its image contains no nontrivial scalar matrices. Moreover whenever $\omega \notin G$ then W is not contained in an eigenspace of $\pi \circ \Phi_f(\omega)$.

Proof. Let $\omega \in G * H$ and write it as a reduced word $\omega = h_1 g_1 \cdot \dots \cdot h_k g_k$, with $g_i \in G, h_i \in H$, all of them nontrivial except possibly g_k and h_1 . We may (and shall) also assume that $\omega \notin G$ since in this case there is nothing to prove. Set

$$\mathcal{U}(\omega) = \{f \in \mathrm{SL}_n(\mathbf{R}) : \pi\Phi_f(\omega) \text{ is scalar on } W\}, \quad \mathcal{U} = \bigcap_{\omega \in G * H \setminus G} \mathcal{U}(\omega).$$

By Lemma 3.1 $\mathcal{U}(\omega)$ is Zariski open in SL_n . Indeed if $\dim(W^+) = r$ and $\{w_0, w_1, \dots, w_r\}$ are $r + 1$ vectors in general position in W^+ , the complement is characterized by the equations $\{\pi\Phi_f(\omega)w_i \wedge w_i = 0\}_{0 \leq i \leq r}$. Since $\mathrm{SL}_n(\mathbf{R})$ is connected, if we show that $\mathcal{U}(\omega)$ is nonempty it will follow from the Baire category theorem that \mathcal{U} is nonempty and the theorem will be proved.

Choose a basis $\{v_1, v_2, \dots, v_n\}$ for \mathbf{R}^n a vector $v \in W$ and a number $L \gg 0$. Let $a^+ = [v_1]$, $a^- = [v_2]$, $x = [v]$, $H^+ = [\mathrm{Span}\{v_2, v_3, \dots, v_n\}]$ and $H^- = [\mathrm{Span}\{v_1, v_3, \dots, v_n\}]$ three projective points and two projective hyperplanes corresponding to these vectors. Define $f = f(L) \in \mathrm{SL}_n(\mathbf{Z})$ by the requirements:

$$fv_1 = Lv_1, \quad fv_2 = \frac{1}{L}v_2, \quad fv_i = v_i, \quad \forall 3 \leq i \leq n.$$

The dynamics of f on \mathbb{P} is proximal in the sense that $\lim_{L \rightarrow \infty} f(L)^{\pm 1}(x) = a^{\pm}$, $\forall x \notin H^{\pm}$ respectively. Note that

$$\pi\Phi_f(\omega)(x) = \pi f h_1 f^{-1} g_1 f h_2 f^{-1} \dots f h_k f^{-1} g_k(x).$$

Consider the sequence of points $f^{-1}g_k(x)$, $f h_k f^{-1}g_k(x)$, up to $f h_1 f^{-1}g_1 f \dots f h_k f^{-1}g_k(x)$. Our strategy is to choose all of the above data in such a way that these points alternate, coming arbitrarily close to a^- and a^+ respectively when $L \rightarrow \infty$. If we insist also that $x \neq \pi(a^+)$ (or that $x \neq \pi g_1(a^+)$ when $h_1 = 1$) it will follow that when L is large enough $\pi\Phi_f(\omega)$ does not fix the projective point $x \in W$ and the claim will follow. To carry this out we require the following:

- (1) $g_k(x) \notin H^-$ (in particular $x \notin H^-$ if it so happens that $g_k = 1$)
- (2) $g_i(a^+) \notin H^-$, for $1 \leq i < k$
- (3) $h_i(a^-) \notin H^+$, for $1 < i \leq k$
- (4) $\pi(a^+) \neq x$, if $h_1 \neq 1$ and $\pi g_1(a^+) \neq x$ otherwise

Since $\{g_1, g_2, \dots, g_{k-1}\}$ are by assumption non scalar matrices we can choose v_1 which is not an eigenvector of any of them. v_2 is chosen to be linearly independent of v_1 and not an eigenvector of any of the matrices $\{h_2, \dots, h_k\}$. Now choose the rest of the basis $\{v_3, \dots, v_n\}$ in such a way that H^{\pm} stay away from the finitely many points as required in the second and third conditions. In addition we choose these hyperplanes in such

a way that $g_k([W^+]) \not\subset H^+$ in order to make the first condition possible. Now choose $v \in W$ so as to actually satisfy the first and fourth conditions. The latter is always possible by our assumption that $\dim(W) > \dim(\ker(\pi)) + 1 = \dim \pi^{-1}(\text{Span}\{v_1\}) = \dim \pi^{-1}(\text{Span}\{g_1 v_1\})$.

It is clear by all these choices, arguing by induction on m , that

$$\lim_{L \rightarrow \infty} \pi \Phi_f(L)(\omega)(x) = \begin{cases} \pi(a^+) \neq x, & \text{if } h_1 \neq 1 \\ \pi g_1(a^+), & \text{if } h_1 = 1 \end{cases} \neq x$$

Which concludes the proof. \square

Remark 4.2. *Similar linearity statements for free products are well known and established for example in [Sha79]. However since we didn't find the exact statement we needed we included the complete proof.*

The previous theorem gives rise to our linear version of Theorem 2.3

Theorem 4.3. *(Free product) Under the assumptions of Theorem 2.1 set $G_1 = G * \mathbf{Z}$ where the group $\mathbf{Z} = \langle f \rangle$. Then there exists an element of infinite order $\ell \in \text{SL}_n(\mathbf{C})$ such that the natural map $\Phi_\ell : G_1 \rightarrow \text{SL}_n(\mathbf{R})$ defined by $g \mapsto g, \forall g \in G$ and $f \mapsto \ell$ is an isomorphism onto its image $\langle G, \ell \rangle < \text{SL}_n(\mathbf{R})$. Moreover all the involutions in G_1 are conjugate, and the only elements of G_1 such that $W^+(t)$ is contained in an eigenspace of $\pi_t \circ \Phi_\ell(g)$ are $g = 1$ and $g = t$.*

Proof. (Of Theorem 4.3) Let H be any infinite cyclic group in $\text{SL}_n(\mathbf{R})$ and $G < \text{SL}_n(\mathbf{R})$ the given group. Applying Proposition 4.1 to these groups G, H yields everything we need leaving only the verification of the fact that all involutions in $G * H$ are conjugate. Let $\sigma \in G * \mathbf{Z}$ be any involution. Being an element of finite order it must stabilize a vertex in the Bass-Serre tree. Since \mathbf{Z} contains no involutions σ is conjugate into G and by our assumption all involutions in G are already conjugate. \square

And here is our linear version of Theorem 2.4

Theorem 4.4. *(HNN extension) Under the assumptions of Theorem 2.1 let $t \neq v \in G$ be an involution and set $G_1 = \langle G, f \mid f^{-1}tf = v \rangle$ and $A_1 := \langle A, f \rangle$. There exists an element of infinite order $\ell \in \text{SL}_n(\mathbf{C})$ such that $\ell^{-1}t\ell = v$ and such that the natural map $\Phi_\ell : G_1 \rightarrow \text{SL}_n(\mathbf{R})$ defined by $g \mapsto g, \forall g \in G$ and $f \mapsto \ell$ is an isomorphism onto its image. Moreover all the involutions in G_1 are conjugate and the only elements of G_1 such that $W^+(t)$ is contained in an eigenspace of $\pi_t \circ g$ are $g = 1$ and $g = t$.*

Proof. (Of Theorem 4.4). Assume that we are in the setting of that theorem and $t, v \in G$ are the two given involutions. Conjugating G , if necessary, we may assume that $t = \text{Diag}(1, \dots, 1, -1, \dots, -1)$ is a diagonal matrix with r ones and $n - r$ minus-ones along the diagonal. If e_1, \dots, e_n denote the vectors of the standard basis then $W^+ :=$

$W^+(t) = \langle e_1, e_2, \dots, e_r \rangle$, and $W^- := W^-(t) = \langle e_{r+1}, e_{r+2}, \dots, e_n \rangle$ are, respectively, the ± 1 eigenspaces of t . By our assumption $r > n - r$ which immediately implies also that $r \geq 2$.

Recall that $G_1 = \langle G, f \mid f^{-1}tf = v \rangle$, let $G_2 = \langle G, k \mid k^{-1}tk = t \rangle$. It follows from a standard argument involving the universal properties of these two HNN extensions that they are isomorphic. Indeed by assumption all the involutions in G are already conjugate within G so there is an element $h \in G$ such that $h^{-1}th = v$. Let $F : G_1 \rightarrow G_2$ be the homomorphism defined by the requirement that $F(g) = g, \forall g \in G, F(f) = kh$ and let $I : G_2 \rightarrow G_1$ be the homomorphism defined by the requirement $I(g) = g, \forall g \in G, I(k) = fh^{-1}$. Since $I \circ F$ fixes pointwise both G and f it must be the identity of G_1 and similarly $F \circ I$ is the identity of G_2 . We will hence identify these two groups and in particular we will identify G_1 as the HNN extension $G_1 = \langle G, k \mid k^{-1}tk = t \rangle$ where $k = fh^{-1}$.

Let $Z = \text{SL}(W^+) \times \text{SL}(W^-) < C_{\text{SL}_n(\mathbf{R})}(t)$. This group is isomorphic to $\text{SL}_r(\mathbf{R}) \times \text{SL}_{n-r}(\mathbf{R})$ and in particular it is a connected closed subgroup of $\text{SL}_n(\mathbf{R})$. For any $u \in Z$ we obtain a homomorphism $\Phi_u : G_1 \rightarrow \text{SL}_n(\mathbf{R})$ given by $g \mapsto g, \forall g \in G$ and $k \mapsto u$.

Our goal is to find some $u \in Z$ such that for every $w \in G_1$, $w \notin \{1, t\}$ the element $\Phi_u(w)$ does not fix $[W^+]$ pointwise. This will show that $\Phi_u : G_1 \rightarrow \langle G, u \rangle$ is an isomorphism and at the same time it will establish the auxiliary requirement that the only elements of G_1 to fix $[W^+]$ pointwise be 1 and t .

Let $w \in G_1 \setminus \{1, t\}$. Let $\{v_0, v_1, \dots, v_r\}$ be any $r+1$ vectors in general position within W^+ . By Lemma 3.1 we see that

$$\begin{aligned} \mathcal{U}_w &:= \{u \in Z \mid W^+ \text{ is not contained in an eigenspace of } \pi_t \circ \Phi_u(w)\} \\ &= Z \setminus \left(\bigcap_{i=0}^r \{u \in Z \mid (\pi_t \circ g(v_i)) \wedge v_i = 0\} \right) \end{aligned}$$

is the complement of a closed subvariety of Z . If we can show that \mathcal{U}_w is nonempty it will follow from the Baire category theorem that $\mathcal{U} = \bigcap_{w \in G_1 \setminus \{1, t\}} \mathcal{U}_w$ is a dense G_δ subset of Z . In particular we would have found some $u \in Z$ satisfying all of our requirements.

From here on we will fix $w \in G_1 \setminus \{1, t\}$ and write this element in a reduced canonical form as $w = g_1 k^{\delta_1} g_2 \dots k^{\delta_m} g_{m+1}$ where $g_i \in G, \delta_i \in \{\pm 1\}$. That the word is reduced means that it is subject to the additional restrictions that $g_i \neq 1$, for $i \in \{2, \dots, m\}$, and that $k^{\delta_i} g_{i+1} k^{\delta_{i+1}}$ is neither of the form $k^{-1}tk$ nor of the form ktk^{-1} for any $i \in \{1 \dots m\}$. Thus

$$\Phi_u(w) = g_1 u^{\delta_1} g_2 \dots u^{\delta_m} g_{m+1}.$$

Let $S = \{g_1, g_1^{-1}, g_2, g_2^{-1}, \dots, g_{m+1}^{-1}\}$ and set $S_0 = S \setminus \{1, t\}$.

By our assumption the set

$$\Omega := \{v \in W^+ \mid v \text{ is not an eigenvector of } \pi_t \circ g \text{ for any } g \in S_0\}$$

is the complement of the union of finitely many proper linear subspaces of W^+ . In particular Ω is a dense open subset of the vector space W^+ . Let $v_1 \in \Omega$ be any vector and $v_2 \in \Omega$ a linearly independent vector that also avoids the finitely many one dimensional subspaces $\{\text{Span}\{\pi_t \circ g(v_1)\} \mid g \in S_0\}$. Since S_0 is symmetric, these choices ensure that $\pi_t \circ g(v_i)$ and v_j are linearly independent for every $i, j \in \{1, 2\}$. Thus we may complete this pair of vectors to a basis $\{v_1, v_2, \dots, v_r\}$ for W^+ in such a way that the two codimension one spaces $\text{Span}\{v_1, v_3, v_4, \dots, v_r\}$ and $\text{Span}\{v_2, v_3, \dots, v_r\}$ of W^+ do not contain any of the points $\{\pi_t \circ g(v_i) \mid g \in S_0, i \in \{1, 2\}\}$. Finally let $\{v_{r+1}, v_{r+2}, \dots, v_n\}$ be a basis for W^- . So that $\{v_1, v_2, \dots, v_n\}$ becomes a basis for the whole space which is compatible with the direct sum decomposition $W^+ \oplus W^-$. Let $a^+ = [v_1]$, $a^- = [v_2]$, $H^+ = [\text{Span}\{v_2, v_3, \dots, v_n\}]$ and $H^- = [\text{Span}\{v_1, v_3, \dots, v_n\}]$ be two projective points and two projective hyperplanes corresponding to these vectors. Our choices above imply that

$$g(a^\pm) \notin H^- \cup H^+, \quad \forall g \in S_0. \quad (1)$$

Pick a number $L \gg 0$ and define $u = u(L) \in Z$ by the requirements:

$$uv_1 = Lv_1, \quad uv_2 = \frac{1}{L}v_2, \quad uv_i = v_i, \quad \forall 3 \leq i \leq n.$$

The dynamics of the element u on the projective plane \mathbb{P} is very proximal in the sense that

$$\lim_{L \rightarrow \infty} u(L)^{\pm 1}(x) = a^\pm, \quad \forall x \notin H^\pm \text{ respectively.} \quad (2)$$

We denote by $g(L) = \Phi_{u(L)}(w)$ in order to stress the dependence of this element on L .

If $m = 0$ then $g = g_1$ and as long as $g \notin \{1, t\}$ any element $x \in \Omega$ will satisfy $\pi_t \circ g(x) \neq x$ as required. So let us assume that $m \geq 1$ and choose $x \in [W^+]$ to be any projective point such that $g_{m+1}(x) \notin H^+ \cup H^-$ and $g_1(a^\pm) \notin [\pi_t^{-1}(x)]$. Such a choice is possible, regardless of the value of g_1 in view of our assumption that $\dim W^+ > \dim W^-$.

Now argue by induction on m that $\lim_{L \rightarrow \infty} g_1^{-1}g(L)(x) = a^{\delta_1}$. If $m = 1$, $g_1^{-1}g(L) = u^{\delta_1}g_2$. Since by our choice of x , $g_2 \cdot x \notin H^+ \cup H^-$ the desired property follows directly from Equation (2). Now applying the induction hypothesis to the word $\tilde{w} = g_2k_2^{\delta_2}g_3 \dots k_m^{\delta_m}g_{m+1}$ and $\tilde{g}(L) = \Phi_{u(L)}(\tilde{w})$ we obtain

$$\lim_{L \rightarrow \infty} g_1^{-1}g(L)(x) = \lim_{L \rightarrow \infty} u(L)^{\delta_1}g_2g_2^{-1}\tilde{g}(L)(x) = \lim_{L \rightarrow \infty} u(L)^{\delta_1}g_2a^{\delta_2} = a^{\delta_1}$$

The equality before last, uses the induction hypothesis. The last equality is justified as follows. If $g_2 \neq t$ then by Equation (1) $g_2a^{\delta_2} \notin H^+ \cup H^-$, while if $g_2 = t$ then by the fact that the word w is reduced we know that $\delta_1 = \delta_2$ and hence $a^{\delta_2} \notin H^{\delta_1}$. In both cases the claim follows directly from Equation (2). This completes the proof of the induction. By the Baire theoretic argument we have thus found $u \in Z$ such that $\Phi_u : G_1 \rightarrow \text{SL}_n(\mathbf{R})$ becomes an isomorphism and such that the induction hypothesis, according to which W^+ is not contained in an eigenspace of $\Phi_u(w)$ for any $w \in G_1 \setminus \{1, k\}$, is also satisfied.

It remains only to show that all the involutions in G_1 are conjugate. We argue on the Bass-Serre tree corresponding to the HNN extension $G_1 = \langle G, f \mid f^{-1}tf = v \rangle$. If $\sigma \in G_1$ is an involution then, being an element of finite order, it must fix a vertex in the Bass-Serre tree. Since the action of G_1 is transitive on the vertices all vertex stabilizers are conjugate to G . But by the induction hypothesis all involutions in G itself are conjugate and the theorem is proved. \square

5. CONCLUSION

Proof. (of Theorem 1.4) Let $H \curvearrowright X$ be the given action, fix any basepoint $x \in X$ and let $A = H_x$ be its stabilizer. The condition that the action on (ordered) pairs of distinct points is free is equivalent to the condition that A be malnormal in H . We fix any involution $t \in SL_n(\mathbf{R})$ and denote by W^\pm its ± 1 eigenspaces and by $\pi = \mathbf{R}^n \rightarrow W^+$ the projection on W^+ along W^- . We impose also the restriction $\dim(W^+) \geq \dim(W^-) + 2$. Applying Proposition 4.1 to the groups H and $\langle t \rangle$ and then replacing if necessary the given group H by its conjugate obtained in that theorem we may assume that $G := \langle H, t \rangle \cong H * \mathbf{Z}/2\mathbf{Z}$ and that furthermore: (i) A is malnormal in G , (ii) all involutions in G are conjugate to t and (iii) if W^+ is contained in an eigenspace of some $g \in G$ then $G \in \{1, t\}$. Indeed (i),(ii) are guaranteed by the properties of free products and (iii) follows from Proposition 4.1.

The conditions (i),(ii),(iii) are exactly these needed in order to apply Theorem 2.1. Let us enumerate the elements of the group G as follows $G = \{v_1, v_2, \dots\}$. Applying this theorem inductively to obtain a sequence of extensions $(A, G) < (A_1, G_1) < (A_2, G_2) < \dots$, such that $G_i \cap A_{i+1} = A_i$. These come together with elements $f_i \in A_i$ such that $A_i t f_i = A_i v_i$. Let $G_\omega = \bigcup_{i=1}^\infty G_i$ and $A_\omega = \bigcup_{i=1}^\infty A_i$. These groups satisfy the properties (i),(ii),(iii). In addition $G \cap A_\omega = A$ and for every $v \in G$ there exists some $f \in A$ such that $Atf = Av$.

Applying the procedure of the previous paragraph to (A_ω, G_ω) and continuing inductively we obtain a sequence $(A_\omega, G_\omega) < (A_{2\omega}, G_{2\omega}) < \dots$ and the union $H_1 = G_{\omega \cdot \omega} = \bigcup_{i=1}^\infty G_{i\omega}$ and $B_1 = A_{\omega \cdot \omega} = \bigcup_{i=1}^\infty A_{i\omega}$ satisfy the following conditions:

- B_1 is malnormal in H_1
- For every $v \in H_1$ there exists an element $f \in B_1$ such that $B_1 t f = B_1 v$
- $H \cap B_1 = A$

The first two properties are respectively equivalent to the action $H_1 \curvearrowright B_1 \setminus H_1$ being free and transitive on ordered pairs of distinct points. The last property implies that the action of H on $B_1 \cdot H$ is isomorphic to the original given action $H \curvearrowright A \setminus H$.

We know that $p\text{-char}(H_1) = 2$ because the stabilizer of a point is B_1 and this group contains no involutions by construction. If $M \triangleleft H_1$ is a normal Abelian subgroup then $M \cap G_\alpha$ is a normal abelian subgroup of G_α for every ordinal α and by construction these

groups clearly have non nontrivial normal abelian subgroups. Thus M itself must be trivial. This proves that H_1 does not split. A similar argument shows that if $M, N \triangleleft H_1$ are two commuting normal subgroups then one of them must be trivial.

Finally we appeal to the main theorem of [GG13] (see also [GG08]) to deduce that H_1 admits uncountably many non-isomorphic primitive permutation representations. We have to verify that H_1 is not of Affine or diagonal type. And this immediately follows from the fact that H_1 contains neither normal abelian subgroups, nor pairs of commuting normal subgroups respectively. \square

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